Pulse vaccination in the periodic infection rate SIR epidemic model

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A pulse vaccination SIR model with periodic infection rate $\beta(t)$ have been proposed and studied. The basic reproductive number R_0 is defined. The dynamical behaviors of the model are analyzed with the help of persistence, bifurcation and global stability. It has been shown that the infection-free periodic solution is globally stable provided $R_0 < 1$ and is unstable if $R_0 > 1$. Standard bifurcation theory have been used to show the existence of the positive periodic solution for the case of $R_0 \to 1^+$. Finally, the numerical simulations have been performed to show the uniqueness and the global stability of the positive periodic solution of the system.

Key words: Epidemic model; Basic reproductive number; Pulse vaccination; Uniform Persistence; Globally stability; Infection-free periodic solution; Positive periodic solution.

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1. Introduction

Transmissible diseases have tremendous influence in human life. Every year billions of people suffer or die in various infectious diseases. In recent times, the emerging and reemerging communicable diseases have led to a revive interest in the study of infectious disease. Mathematical models are widely used to understand the mechanisms of spread of infectious diseases, and become an important tool to analyze the spreading and controlling the diseases [1]. Epidemiology modelling and mathematical study on those model can contribute to the design and analysis of epidemiological surveys, suggest some crucial conclusion that should be collected. By determining the key parameters, and finding it's effects of changes in the parameter values, it identify the trends, make general forecasts, and finally estimate the uncertainty in forecasts [2]. Another important motivation of the development of this field is the evaluation of various vaccination/control strategies for human as well as animal. Vaccination is an effective way to control the transmission of a disease among the species. Mathematical modelling can contribute to the design and assessment of the vaccination strategies. A theoretical examination of pulse vaccination policy for SIR model have been shown by Stone et. al [3]. They found a disease-free periodic solution with the same period as the pulse vaccination, and studied the local stability of this solution. Nokes and Swinton have studied the control of childhood viral infections by pulse vaccination [4]. Major application of the pulse vaccination method for SIR and SEIR epidemic models have been studied by A.d'onofrio and his group [5-7,9]. They have studied global asymptotic eradication in the presence of vaccine failure [5]. Yicang zhou and Hanwu Liu studied the stability of Periodic Solutions for an SIS Model with Pulse Vaccination [8]. Fuhrman et. al studied asymptotic behavior of an SI Epidemic Model with Pulse Removal [10]. The global stability of the disease-free periodic solution for SIR and SIRS models with Pulse Vaccination have been investigated by Jin [11].

Recently, pulse vaccination, the repeated application of vaccine over a defined age range, is gained prominence as a strategy for the elimination of childhood viral infectious such as measles and poliomyelitis. This policy is based on the suggestion that measles epidemics can be more efficiently controlled when the natural temporal process of the epidemics is antagonized by another temporal process [12,13].

On the other hand , in the last 30 years epidemiology of infectious diseases has focused on the theoretical and the numerical-experimental study of the effects of periodically varying external factors on the time course of the incidence of infectious diseases. It is well known that the epidemic models with constant contact rate are not able to reproduce realistic endemic situations, since incidence of many infectious diseases are not constant [14]. The endemic nature of many communicable diseases is characterized by a wide range of temporal oscillatory patterns: annual or polyannual periodicity [15], apparently random oscillations with a strong annual component in the power spectrum [16] etc.

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In this paper, we have studied the dynamical behavior of a pulse vaccination SIR model with periodic infection rate. The paper is organized as follows: Section 2 gives the SIR model with pulse vaccination and definition of the basic reproductive number R_0, and demonstrates the existence of an infection-free periodic solution. The global stability of the infection-free periodic solution is obtained in Section 3. In Section 4, we discuss the persistence of the infectious disease by obtaining the uniform persistence conditions $R_0 > 1$ of the infectious disease. Section 5, concentrates on the existence of the positive periodic solution of the periodic SIR model with pulse vaccination. The existence of positive periodic solution is obtained by using a well-known bifurcation result of Rabinowitz [17] for $R_0 \to 1^+$. Finally, numerical simulations and a brief discussion conclude the paper.

2. The SIR model without and With pulse vaccination

Our SIR model is based on the following assumptions:

1. The total population size is constant and is denoted by N, with $N = S + I + R$, the population is divided into three groups: (a) The susceptible class, S, comprising those people who are capable of catching the disease; (b) The infectives, I, comprising those who are infected and capable of transmitting the disease; (c) The recovered class, R, comprising those individuals who are immune.

2. The total population size is N and the per capita birth rate is a constant b. As births balance deaths we must have that the per capita death rate is also b.

3. The population is uniform and mixes homogeneously.

4. The infection rate $\beta(t)$ is defined as the total rate at which potentially infectious contacts occur between two individuals. A potentially infectious contact is one which will transmit the disease if one individual is susceptible and the other is infectious, so the total rate at which susceptibles become infected is $\beta(t)SI$: Biological considerations mean that $\beta(t)$ is a continuous function. We also assume that $\beta(t)$ is not identically zero, positive, non-constant and periodic function with period $\omega > 0$.

5. The infectives move from the infectives class to the recovered class at a constant rate γ where $(1/\gamma)$ is the average infectious period conditional on survival to the end of it.

6. We also assume that α is a non-negative constant and represents the death rate due to the disease (that is, the disease-related death rate).

On the basis of the above assumptions a SIR model can be written as a set of coupled non-linear ordinary differential equations as follows:

$$\begin{cases} S' = bK - \beta(t)SI - bS, \\ I' = \beta(t)SI - (b + \gamma + \alpha)I, \\ R' = \gamma I - bR. \end{cases} \tag{2.1}$$

Here the parameters K, b, γ are all positive constants. The constant $bK \equiv A$ is the inflow rates, b is the per capita birth rate and the per capita death rate, so that K represents a carrying capacity, or maximum possible population size.

Adding the three equation of system (2.1), we get

$$N'(t) = bK - bN - \alpha I. \tag{2.2}$$

For small value of the parameter α the total population size become closer to the carrying capacity K.

According to usual convention, pulse vaccination can be defined as the repeated application of vaccine across an age range. Let us assume the pulse scheme proposes to vaccinate a fraction p_k of the entire susceptible population in a single pulse, applied k time in time period $[0, \omega]$. Pulse vaccination gives lifelong immunity to $p_k S$ susceptibles who are, as a consequence, transferred to the recovered class (R) of the population.

When pulse vaccination is incorporated to SIR model (2.1), the model takes the form as follows:

$$\begin{cases} S' = bK - \beta(t)SI - bS, & t \neq \tau_k. \\ I' = \beta(t)SI - (b + \gamma + \alpha)I, & k = 0, 1, 2 \cdots. \\ R' = \gamma I - bR, \end{cases} \tag{2.3}$$

$$\begin{cases} S(\tau_k^+) = (1 - p_k)S(\tau_k), \\ I(\tau_k^+) = I(\tau_k), & k = 0, 1, 2 \cdots. \\ R(\tau_k^+) = R(\tau_k) + p_k S(\tau_k). \end{cases}, \tag{2.4}$$

where $0 \leq p_k < 1 (k \in Z_+)$ are constants and $q > 0$ is an integer such that $p_{k+q} = p_k, \tau_{k+q} = \tau_k + \omega$. Note that the dynamics of the total population size $N(t)$ still satisfy equation (2.2).

From biological view point, we can assume that the domain

$$\Omega = \{(S, I, R) : S \geq 0, I \geq 0, R \geq 0, S + I + R < K\}$$

is a positive invariant set of system (2.3) and (2.4).

Let us start to analyze system (2.3) and (2.4) by demonstrating the existence of an infection-free periodic solution, in which infectious individuals are entirely absent from the population permanently i.e.,

$$I(t) = 0, t \geq 0. \tag{2.5}$$

Under this conditions, the systems (2.3) and (2.4) reduce to

$$\begin{cases} S' = bK - bS, & t \neq \tau_k. \\ R' = -bR, & k = 0, 1, 2 \cdots . \\ S(\tau_k^+) = (1 - p_k)S(\tau_k), \\ R(\tau_k^+) = R(\tau_k) + p_k S(\tau_k). \end{cases} \tag{2.6}$$

and $N' = bK - bN$, therefore $N(t) \to K$ as $t \to \infty$. For convenient, we take $N = S + R = K$. Then, solving the equation (2.6), we get

$$S(t) = W(t, 0)S(0) + bK \int_0^t W(t, \tau)d\tau$$

and

$$R(t) = W(t, 0)R(0) + \Sigma_{0 < \tau_k < t} W(t, \tau_k)p_k,$$

where

$$W(t, \tau) = \prod_{\tau < \tau_j < t} (1 - p_j)e^{-b(t-\tau)}.$$

Since $W(\omega, 0) = \prod_{j=1}^{q} (1 - p_j)e^{-b\omega} < 1$, therefore equation (2.6) has a unique ω-periodic solution $(S^*(t), 0, R^*(t))$ with the initial conditions $S^*(0) = bK \int_0^\omega W(\omega, \tau)d\tau / (1 - W(\omega, 0))$, $R^*(0) = \sum_{0 < \tau_k < t} W(\omega, \tau_k)/(1 - W(\omega, 0))$.

Let us define the basic reproductive rate of model (2.3) and (2.4) as follows:

$$R_0 = \frac{\int_0^\omega \beta(t)S^*(t)dt}{\omega(b + \alpha + \gamma)},$$

where $S^*(t)$ is the periodic infection-free solution.

3. Local and global asymptotic stability of the infection free solution

In this section, we will prove the local and global asymptotic stability of the infection free solution $(S^*(t), 0, R^*(t))$.

The local stability of the ω-period solution $(S^*(t), 0, R^*(t))$ may be determined by considering the linearized SIR equation of (2.3), (2.4) about the known periodic solution $(S^*(t), 0, R^*(t))$ by setting $S(t) = S^*(t) + x(t)$, $I(t) = y(t)$, $R(t) = R^*(t) + z(t)$, where $x(t), y(t)$, and $z(t)$ are small perturbation. The linearized equations may be written as

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix},$$

where $\Phi(t) = \varphi_{ij}(t), i, j = 1, 2, 3$ satisfies

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} -b & -\beta(t)S^*(t) & 0 \\ 0 & \beta(t)S^*(t) - (b + \gamma + \alpha) & 0 \\ 0 & \gamma & -b \end{pmatrix} \Phi(t),$$

with $\Phi(0) = E$, where E is the identity matrix. Therefore, the system (2.4) become

$$
\begin{pmatrix} x(\tau_k^+) \\ y(\tau_k^+) \\ z(\tau_k^+) \end{pmatrix} = \begin{pmatrix} 1 - p_k & 0 & 0 \\ 0 & 1 & 0 \\ p_k & 0 & 1 \end{pmatrix} \begin{pmatrix} x(\tau_k) \\ y(\tau_k) \\ z(\tau_k) \end{pmatrix}.
$$

Hence, according to the Floquet theory, if all eigenvalues of

$$
M = \begin{pmatrix} \prod\limits_{k=1}^{q} (1 - p_k) & 0 & 0 \\ 0 & 1 & 0 \\ p_q \prod\limits_{k=2}^{q} (1 - p_k) & 0 & 1 \end{pmatrix} \Phi(\omega)
$$

are less than one, then the ω-periodic solution $(S^*(t), 0, R^*(t))$ is locally stable. After calculation we get,

$$
\Phi(t) = \begin{pmatrix} e^{-bt} & \varphi_{12} & 0 \\ 0 & \varphi_{22} & 0 \\ 0 & \varphi_{32} & e^{-bt} \end{pmatrix},
$$

where

$$
\begin{aligned}
\varphi_{22} &= \exp(\int_0^t [\beta(\tau)S^*(\tau) - (b + \alpha + \gamma]d\tau) \\
\varphi_{12} &= -\exp(-bt)\int_0^t \beta(\tau)S^*(\tau)\varphi_{22}(\tau)e^{b\tau}d\tau, \\
\varphi_{32} &= \exp -bt \int_0^t \gamma\varphi_{22}(\tau)e^{b\tau}d\tau.
\end{aligned}
$$

Thus, if μ_1, μ_2 and μ_3 are the eigenvalues of matrix M, then they are given by,

$$
\mu_1 = \prod_{k=1}^{q} (1 - p_k)e^{-b\omega}, \mu_3 = e^{-b\omega}
$$

and

$$
\mu_2 = \varphi_{22}(\omega) = \exp(\int_0^t \beta(\tau)S^*(\tau)d\tau - (b + \alpha + \gamma)\omega.
$$

Obviously $\mu_1 < 1$, $\mu_3 < 1$ and $\mu_2 < 1$ if and only if

$$
\frac{\int_0^\omega \beta(\tau)S^*(\tau)d\tau}{\omega} < b + \alpha + \gamma.
$$

Therefore, the periodic infection-free solution $(S^*(t), 0, R^*(t))$ is asymptotically stable if $R_0 < 1$. Now we are in a position to summarize the above results in the following theorem:

Theorem 3.1. The periodic infection-free solution $(S^*(t), 0, R^*(t))$ of system (2.3) and (2.4) is local asymptotically stable provided $R_0 < 1$.

In order to prove the global stability of infection-free solution $(S^*(t), 0, R^*(t))$, we need the following lemma.

Lemma 3.1. (Comparison theory [18]). Assume that $m \in PC[R_+, R]$ with points of discontinuity at $t = \tau_n$ and is left continuous at $t = \tau_n$, $n = 1, 2, \cdots$, and

$$
\begin{cases} D_-m(t) \leq g(t, m(t)), t \neq \tau_n, n = 1, 2, \cdots, \\ m(\tau_n^+) \leq \psi_n(m(\tau_n)), t = \tau_n, n = 1, 2, \cdots, \end{cases} \tag{3.1}
$$

where $g \in C[R_+ \times R_+, R]$, $\psi_n \in C[R, R]$ and $\psi_n(u)$ is nondecreasing in u for each $n = 1, 2, \cdots$. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$
\begin{cases} u' = g(t, m(t)), t \neq \tau_n, n = 1, 2, \cdots, \\ u(\tau_n^+) = \psi_n(u(\tau_n)), t = \tau_n, n = 1, 2, \cdots, \\ u(t_0^+) = u_0, \end{cases} \tag{3.2}
$$

existing on $[t_0, \infty)$. Then $m(t_0^+) \le u_0$ implies $m(t) \le r(t)$.

Remark. In Lemma 3.1, assume the inequalities (3.1) reversed. Let $\rho(t)$ be the minimal solution of (3.2) existing on $[t_0, \infty)$. Then, $m(t) \ge \rho(t)$.

Lemma 3.2. For the following impulsive equation

$$\begin{cases} S' = bK - bS, & t \ne \tau_k. \\ S(\tau_k^+) = (1 - p_k)S(\tau_k), & n = 1, 2, \cdots, \end{cases} \tag{3.3}$$

has a unique positive ω-periodic solution $S^*(t, S_0^*)$ for which $S^*(0, S_0^*) = S_0^*$ and $S^*(t, S_0^*) > 0, t \in R_+$, and $S^*(t, S_0^*)$ is global asymptotically stable in the sense that $\lim_{t \to \infty} |S(t, S_0) - S^*(t, S_0^*)| = 0$, where $S(t, S_0)$ is any solution of system (3.3) with positive initial value $S(0, S_0) = S_0 > 0$.

Proof. The first part of the proof follows from the results of Section 2. Therefore we have to show only $\lim_{t \to \infty} |S(t, S_0) - S^*(t, S_0^*)| = 0$.

Since

$$|S(t, S_0) - S^*(t, S_0^*)| = W(t, 0)|S_0 - S_0^*|,$$

thus we need to show $W(t, 0) \to 0$ as $\to \infty$. Suppose $t \in (n\omega, (n+1)\omega]$, then

$$\begin{aligned} W(t, 0) &= \prod_{0 < \tau_j < t} (1 - p_j)e^{-bt} \\ &= \prod_{0 < \tau_j < n\omega} (1 - p_j)e^{-bn\omega} \prod_{n\omega < \tau_j < t} (1 - p_j)e^{-b(t - n\omega)} \\ &= [\prod_{0 < \tau_j < \omega} (1 - p_j)e^{-b\omega}]^n \prod_{n\omega < \tau_j < t} (1 - p_j)e^{-b(t - n\omega)} \\ &\le \prod_{0 < \tau_j < \omega} (1 - p_j)[\prod_{0 < \tau_j < \omega} (1 - p_j)e^{-b\omega}]^n \end{aligned}$$

Thus $\lim_{t \to \infty} W(t, 0) = 0$, since $\prod_{0 < \tau_j < \omega} (1 - p_j)e^{-b\omega} < 1$. Hence the proof.

Now, we are totally ready to show the global asymptotic stability conditions of the infection free solution. We claim the following theorem:

Theorem 3.2. The periodic infection-free solution $(S^*(t), 0, R^*(t))$ of system (2.3) and (2.4) is globally stable if $R_0 < 1$.

Proof. From the first equation of (2.3) and (2.4), and by using Lemma 3.1 and 3.2, we obtain that, for any given $0 < \varepsilon < \omega(1 - R_0)(b + \gamma + \alpha)/2 \int_0^\omega \beta(t)dt$, there exists $T_1 > 0$, such that

$$S(t) < S^*(t) + \varepsilon, \quad \forall t > T_1. \tag{3.4}$$

By substituting (3.4) into second equation of (2.3), we obtain

$$\frac{dI(t)}{dt} = \beta(t)SI - (b + \gamma + \alpha)I \le \beta(t)S^*(t)I(t) - (b + \gamma + \alpha)I(t) + \varepsilon\beta(t)I(t)$$

By comparison theory, for $t \in (T_1 + n\omega, T_1 + (n+1)\omega]$, we have

$$\begin{aligned} I(t) &\le I(T_1)\exp(\int_{T_1}^t [\beta(\tau)S^*(\tau) - (b + \gamma + \alpha) + \beta(\tau)\varepsilon])d\tau \\ &= I(T_1)\exp\{\int_{T_1}^{T_1+n\omega} [\beta(\tau)S^*(\tau) - (b + \gamma + \alpha)]d\tau + \varepsilon \int_{T_1}^{T_1+n\omega} \beta(\tau)d\tau \\ &\quad + \varepsilon \int_{T_1+n\omega}^t \beta(\tau)d\tau + \int_{T_1+n\omega}^t [\beta(\tau)S^*(\tau) - (b + \gamma + \alpha)]d\tau\} \\ &\le I(T_1)\exp\{n \int_0^\omega [\beta(\tau)S^*(\tau) - (b + \gamma + \alpha)]d\tau + (n+1)\varepsilon \int_0^\omega \beta(\tau)d\tau + \int_0^\omega \beta(\tau)S^*(\tau)d\tau \\ &\le B\exp\{n\omega(b + \gamma + \alpha)(R_0 - 1) + n\varepsilon \int_0^\omega \beta(\tau)d\tau\} \\ &\le B\exp\{n\omega(b + \gamma + \alpha)\tfrac{1}{2}(R_0 - 1)\} \end{aligned} \tag{3.5}$$

where $B = I(T_1)\exp\{\int_0^\omega \beta(\tau)(\varepsilon + S^*(t))d\tau\}$. Thus if $R_0 < 1$, it follows from the above equation that $\lim_{t \to \infty} I(t) = 0$.

Let $H(t) = |S(t) - S^*(t)|$, then we have

$$D_+H(t) = \text{sign}(S(t) - S^*(t))(S'(t) - S^{*'}(t)) \le -bH(t) + \beta(t)SI, \tag{3.6}$$

$$H(\tau_k^+) = (1 - p_k)H(\tau_k).\tag{3.7}$$

Since $S(t) \leq K$, $\beta(t) \leq \max_{t \in [0,\omega]} \beta(t) \equiv \beta^*$, the equation (3.6) reduces to

$$D_+ H(t) \leq -bH(t) + K\beta^* I(t).\tag{3.8}$$

For $t \in (T_1 + n\omega, T_1 + (n+1)\omega]$, from (3.5), we see that $I(t) \leq Be^{-\theta n}$, where $\theta = \frac{1}{2}\omega(b + \gamma + \alpha)(1 - R_0)$. By lemma 3.1 and (3.7), (3.8), we have

$$H(t) \leq H(T_1) \prod_{T_1 < \tau_j < t} (1 - p_j)e^{-b(t - T_1)} + K\beta^* \int_{T_1}^t \prod_{\tau < \tau_j < t} (1 - p_j)e^{-b(t - \tau)} I(\tau)d\tau.\tag{3.9}$$

Since

$$
\begin{aligned}
&\int_{T_1}^t \prod_{\tau < \tau_j < t} (1 - p_j)e^{-b(t - \tau)} I(\tau)d\tau \\
&= \int_{T_1}^{\omega + T_1} \prod_{\tau < \tau_j < t} (1 - p_j)e^{-b(t - \tau)} I(\tau)d\tau + \int_{\omega + T_1}^{2\omega + T_1} \prod_{\tau < \tau_j < t} (1 - p_j)e^{-b(t - \tau)} I(\tau)d\tau + \cdots \\
&+ \int_{(n-1)\omega + T_1}^{n\omega + T_1} \prod_{\tau < \tau_j < t} (1 - p_j)e^{-b(t - \tau)} I(\tau)d\tau + \int_{n\omega + T_1}^t \prod_{\tau < \tau_j < t} (1 - p_j)e^{-b(t - \tau)} I(\tau)d\tau \\
&\leq B[\int_{T_1}^{\omega + T_1} \prod_{\tau < \tau_j < t} (1 - p_j)e^{-b(t - \tau)}d\tau + \int_{\omega + T_1}^{2\omega + T_1} \prod_{\tau < \tau_j < t} (1 - p_j)e^{-b(t - \tau)}e^{-\theta}d\tau + \cdots \\
&+ \int_{(n-1)\omega + T_1}^{n\omega + T_1} \prod_{\tau < \tau_j < t} (1 - p_j)e^{-b(t - \tau)}e^{-(n-1)\theta}d\tau + \int_{n\omega + T_1}^t \prod_{\tau < \tau_j < t} (1 - p_j)e^{-b(t - \tau)}e^{-\theta n}d\tau] \\
&\leq Be^{bT_1}(e^{b\omega} - 1)e^{-bt}[\eta^{n-1} + \eta^{n-2}e^{b\omega - \theta} + \eta^{n-3}e^{2(b\omega - \theta)} + \cdots + \eta e^{(n-1)(b\omega - \theta)} + e^{n(b\omega - \theta)}]/b \\
&= Be^{bT_1}(e^{b\omega} - 1)e^{-bt}[\eta^n - e^{n(b\omega - \theta)}]/b(\eta - e^{(b\omega - \theta)}) \\
&\leq B(e^{b\omega} - 1)e^{-bn\omega}[\eta^n - e^{n(b\omega - \theta)}]/b(\eta - e^{(b\omega - \theta)}),
\end{aligned}
\tag{3.10}
$$

where $0 < \eta = \prod_{j=1}^q (1 - p_j) \leq 1$. From (3.9) and (3.10), we obtain that $\lim_{t \to \infty} H(t) = 0$, i.e., $S(t) \to S^*(t)$ as $t \to \infty$.

Similarly, we can prove $R(t) \to R^*(t)$ as $t \to \infty$. The proof is completed.

4. The uniform persistence of the infectious disease

In this section, we will discuss the uniform persistence of the infectious disease, that is, $\lim_{t \to \infty} \inf I(t) > \alpha$ if $R_0 > 1$.

To discus the uniform persistence, we need a lemma first.

Lemma 4.1. If $R_0 > 1$, then the disease uniformly weakly persists in the population, in the sense that we will able to find a constant $c > 0$ such that $\lim_{t \to \infty} \sup I(t) > c$ for all solutions of (2.3) and (2.4).

Proof. Let us suppose that for a given $\varepsilon > 0$, there exists a solution with $\lim_{t \to \infty} \sup I(t) < \varepsilon$. From the first equation of (2.3), we have

$$S' = bK - \beta(t)SI - bS \geq bK - \beta^* K\varepsilon - bS, t \neq \tau_k.$$

Consider the following equation

$$
\begin{cases}
u' = bK - \beta^* K\varepsilon - bu, t \neq \tau_k. \\
u(\tau_k^+) = (1 - p_k)u(\tau_k), k = 1, 2, \cdots
\end{cases}
\tag{4.1}
$$

By lemma 3.2., the (4.1) has a unique positive ω-periodic solution $u^*(t)$ for which $u^*(0) = u_0^*$ and $u^*(t)$ is global asymptotically stable. So

$$S^*(t) - u^*(t) = \beta^* K\varepsilon\left[\frac{W(t, 0)\int_0^\omega W(\omega, \tau)d\tau}{1 - W(\omega, 0)} + \int_0^t W(t, \tau)d\tau\right].\tag{4.2}$$

Let

$$\Delta = \beta^* K \max_{0 \leq t \leq \omega}\left\{\frac{W(t, 0)\int_0^\omega W(\omega, \tau)d\tau}{1 - W(\omega, 0)} + \int_0^t W(t, \tau)d\tau\right\}.$$

By (4.2), we see that

$$u^*(t) \geq S^*(t) - \Delta\varepsilon. \tag{4.3}$$

.

By comparison theory, we obtain that

$$
\begin{aligned}
I' &= \beta(t)SI - (b + \gamma + \alpha)I \\
&\geq I(t)[\beta(t)u(t) - (b + \gamma + \alpha)].
\end{aligned} \tag{4.4}
$$

Since, $u^*(t)$ is global asymptotically stable, for our previous ε above, there exist $T_1 > 0$, such that $u(t) \geq u^*(t) - \varepsilon$, $t > T_1$. From (4.3) and (4.4), we get

$$I'(t) \geq I(t)[\beta(t)S^*(t) - (b + \gamma + \alpha) - \varepsilon(1 + \Delta)\beta(t]. \tag{4.5}$$

Integrating over intervals $[T_1, t]$ we obtain

$$
\begin{aligned}
I(t) &\geq I(T_1)\exp(\int_{T_1}^{t}[\beta(\tau)S^*(\tau) - (b + \gamma + \alpha) - \varepsilon(1 + \Delta)\beta(t)]d\tau \\
&= I(T_1)\exp\{\int_{T_1}^{T_1+n\omega}[\beta(\tau)S^*(\tau) - (b + \gamma + \alpha)]d\tau - \varepsilon(1 + \Delta)\beta(t)n\omega \\
&\quad + \int_{T_1+n\omega}^{t}[\beta(\tau)S^*(\tau) - (b + \gamma + \alpha)]d\tau - \varepsilon(1 + \Delta)\beta(t)(t - n\omega)\} \\
&\geq C\exp\{n\omega(b + \gamma + \alpha)(R_0 - 1) - \varepsilon(1 + \Delta)\beta^*n\omega\}.
\end{aligned} \tag{4.6}
$$

where $t \in (T_1 + n\omega, T_1 + (n + 1)\omega]$, $C = I(T_1)\exp[-(b + \gamma + \alpha + \varepsilon(1 + \Delta)\beta^*)\omega]$. Taking

$$0 < \varepsilon \leq \frac{(b + \gamma + \alpha)(R_0 - 1)}{2\beta^*(1 + \Delta)},$$

thus $I(t) \to \infty$ as $t \to \infty$, a contradiction to the fact that $I(t)$ is bounded. This finishes the proof.

Theorem 4.1. If $R_0 > 1$, then the disease uniformly persistence, it is that there exists a positive constant σ such that for every positive solution of (2.3) and (2.4),

$$\liminf_{t \to \infty} I(t) \geq \sigma > 0.$$

Proof. Let

$$0 < \eta \leq \frac{bK(R_0 - 1)}{2R_0(\alpha + \gamma)}.$$

It can be obtained from the Lemma 4.1 that for any positive solution of (2.3) and (2.4) there exists at least one $t_0 > 0$ such that $I(t_0) > \eta > 0$. Then, we are left to consider two possibilities. The first case is $I(t) \geq \eta$ for all large $t \geq t_0$. The second one is $I(t)$ oscillates about η for large t. The conclusion of Theorem 4.1 is obvious in the first case since we can choose $\sigma = \eta$.

For the second case, let $t_1 > t_0$ and $t_2 > t_1$ satisfy

$$I(t_1) = I(t_2) = \eta, \quad \text{and} \quad I(t) < \eta \quad \text{for} \quad t_1 < t < t_2.$$

Next, we introduce a new variable $V = S + I$, and it follows from the first two equations of (2.3) and (2.4) that

$$
\begin{cases}
V' = bK - bV - (\gamma + \alpha)I, & t \neq \tau_k. \\
I' = \beta(t)(V - I)I - (b + \gamma + \alpha)I, & k = 0, 1, 2 \cdots.
\end{cases} \tag{4.7}
$$

$$
\begin{cases}
V(\tau_k^+) = (1 - p_k)V(\tau_k) + p_k I(\tau_k), \\
I(\tau_k^+) = I(\tau_k), & k = 0, 1, 2 \cdots.
\end{cases} \tag{4.8}
$$

If $I(t) \leq \eta$, then

$$
\begin{cases}
V' = bK - bV - (\gamma + \alpha)I > \frac{bK}{2}(1 + \frac{1}{R_0}) - bV, & t \neq \tau_k. \\
V(\tau_k^+) > (1 - p_k)V(\tau_k)).
\end{cases} \tag{4.9}
$$

Consider the following equation

$$\begin{cases} x' = \frac{bK}{2}(1 + \frac{1}{R_0}) - bx, & t \neq \tau_k. \\ x(\tau_k^+) = (1 - p_k)x(\tau_k). \end{cases} \tag{4.10}$$

By using Lemma 3.2, we can obtain equation (4.10) has a unique positive ω- periodic solution $x^*(t)$, and $x^*(t)$ is global asymptotically stable in the sense that $\lim_{t\to\infty} |x(t, x_0) - x^*(t)| = 0$, where $x(t, x_0)$ is any solution of system (4.10) with positive initial value $x(0, x_0) = x_0 > 0$. By (3.3) and (4.13) we can easily get

$$x^*(t) = \frac{1}{2}(1 + \frac{1}{R_0})S^*(t)$$

where $S^*(t)$ is a unique positive ω- periodic solution of system (3.3).

Then the comparison principle and the global asymptotically stable of $x^*(t)$ implies that there exists a positive constant $T_1 > 0$, such that

$$V(t) > \frac{1}{2}(1 + \frac{1}{R_0})S^*(t), \quad \text{for all} \quad t > t_1 + T_1. \tag{4.11}$$

For above t_1 and η the solution of the initial value problem

$$y'(t) = \beta(t)[\frac{1}{2}(1 + \frac{1}{R_0})S^*(t) - y]y - (b + \gamma + \alpha)y, \quad y(t_1) = \eta \tag{4.12}$$

is

$$y(t) = \frac{\eta \exp(\int_{t_1}^t [\frac{1}{2}(1 + \frac{1}{R_0})\beta(\tau)S^*(\tau) - (b + \gamma + \alpha)]d\tau)}{1 + \eta \int_{t_1}^t \beta(\tau) \exp(\int_{t_1}^\tau [\frac{1}{2}(1 + \frac{1}{R_0})\beta(\theta)S^*(\theta) - (b + \gamma + \alpha)]d\theta)d\tau} \tag{4.13}$$

The differential equation in (4.12) has a periodic solution

$$y^*(t) = \frac{y^* \exp(\int_{t_1}^t [\frac{1}{2}(1 + \frac{1}{R_0})\beta(\tau)S^*(\tau) - (b + \gamma + \alpha)]d\tau)}{1 + y^* \int_{t_1}^t \beta(\tau) \exp(\int_{t_1}^\tau [\frac{1}{2}(1 + \frac{1}{R_0})\beta(\theta)S^*(\theta) - (b + \gamma + \alpha)]d\theta)d\tau} \tag{4.14}$$

where $y^* = \frac{\exp(\int_{t_1}^{t_1+\omega} [\frac{1}{2}(1+\frac{1}{R_0})\beta(\tau)S^*(\tau) - (b+\gamma+\alpha)]d\tau) - 1}{\int_{t_1}^{t_1+\omega} \beta(\tau) \exp(\int_{t_1}^\tau [\frac{1}{2}(1+\frac{1}{R_0})\beta(\theta)S^*(\theta) - (b+\gamma+\alpha)]d\theta)d\tau}$. Since the condition $R_0 > 1$ of Theorem 4.1 implies that $\int_0^\omega [\frac{1}{2}(1 + \frac{1}{R_0})\beta(\tau)S^*(\tau) - (b + \gamma + \alpha)]d\tau = \frac{\omega(b+\gamma+\alpha)(R_0-1)}{2} > 0$, from the limits

$$\lim_{t\to\infty} \exp(\int_{t_1}^t [\frac{1}{2}(1 + \frac{1}{R_0})\beta(\tau)S^*(\tau) - (b + \gamma + \alpha)]d\tau) = \infty,$$

$$\lim_{t\to\infty} \int_{t_1}^t \beta(\tau) \exp(\int_{t_1}^\tau [\frac{1}{2}(1 + \frac{1}{R_0})\beta(\theta)S^*(\theta) - (b + \gamma + \alpha)]d\theta)d\tau = \infty$$

and the expression of $y(t) - y^*(t)$ it follows that

$$\lim_{t\to\infty} |y(t) - y^*(t)| = 0. \tag{4.15}$$

From (4.15) we see that there exists a positive constant $T_2 > 0$ such that

$$y(t) > \rho \equiv \frac{1}{2} \min_{t_1 \leq t \leq t_1+\omega} y^*(t) > 0, \quad \text{for all} \quad t > t_1 + T_2. \tag{4.16}$$

Let $T = \max\{T_1, T_2\}$ and define

$$\sigma = \min\{\rho, \eta \exp(-(b + \gamma + \alpha)T\}.$$

If $t_2 - t_1 < T$, from the following equation

$$I' = \beta(t)(V - I)I - (b + \gamma + \alpha)I, \tag{4.17}$$

we have the inequality

$$I'(t) > -(b + \gamma + \alpha)I,$$

and the comparison principle implies that $I(t) \geq \eta \exp\{-(b + \gamma + \alpha)\} \geq \eta \exp\{-(b + \gamma + \alpha)T\}$, i.e., $I(t) \geq \sigma$ for all $t \in (t_1, t_2)$. If $t_2 - t_1 > T$, we divide the interval $[t_1, t_2]$ into two subintervals $[t_1, t_1 + T]$ and $[t_1 + T, t_2]$. $I(t) \geq \sigma$ is obvious in the interval $[t_1, t_1 + T]$. By (4.11) and (4.17) we see that in the interval $[t_1 + T, t_2]$,

$$I'(t) \geq \beta(t)[\frac{1}{2}(1 + \frac{1}{R_0})S^*(t) - I]I - (b + \gamma + \alpha)I. \tag{4.18}$$

Therefore again by using comparison principle we get $I(t) \geq y(t) \geq \rho \geq \sigma$ for $t \in [t_1 + T, t_2]$. The above analysis is independent of any interval $[t_1, t_2]$, and the choice of σ is independent of any positive solution of (2.3) and (2.4). Therefore, the persistence is uniform for all positive solution.

5. Existence of the positive ω-periodic solution and bifurcation

Let $PC(J, R)$ $(J \subset R)$ be the set of continuous functions $\psi : J \to R$ for $t \in J, t \neq \tau_k$, that have discontinuities of the first kind at the points $\tau_k \in J$ where they are continuous from the left. Let $PC'(J, R)$ be the set of functions $\psi : J \to R$ with derivative $\frac{d\psi}{dt} \in PC(J, R)$. We consider the Banach spaces of ω-periodic functions $PC_\omega = \{\psi \in PC([0, \omega], R) | \psi(0) = \psi(\omega)\}$ under the supremum norm $\| \psi \|_{PC_\omega} = \sup\{| \psi |: t \in [0, \omega]\}$, and $PC'_\omega = \{\psi \in PC'([0, \omega], R) | \psi(0) = \psi(\omega)\}$ under the supremum norm $\| \psi \|_{PC'_\omega} = \max\{\| \psi \|_{PC_\omega}, \| \psi' \|_{PC'_\omega}\}$. We will also consider the product space $PC_\omega \times PC_\omega$ which is also a Banach space under the norm $\| (\psi_1, \psi_1) \|_{PC_\omega} = \| \psi_1 \|_{PC_\omega} + \| \psi_2 \|_{PC_\omega}$. Moreover, for any $f \in C_\omega$ (or PC_ω) we define the average of f by $\bar{f} := \int_0^\omega f(s)ds/\omega$.

We need the following lemmas which are about linear impulsive periodic equations (also see [20,21]).

Lemma 5.1. Suppose $c_{ij}(t) \in PC_\omega$, and

(1). if $\bar{c}_{11} \neq \frac{1}{\omega} \ln[\prod_{k=1}^q \frac{1}{1-p_k}]$ and $\bar{c}_{22} \neq 0$, then the linear homogeneous periodic impulsive equation

$$\begin{cases} x_1' = c_{11}x_1(t) + c_{12}x_2(t), & t \neq \tau_k \\ x_2' = c_{22}x_2(t), & k = 0, 1, 2 \cdots. \\ x_1(\tau_k^+) = (1 - p_k)x_1(\tau_k), & t = \tau_k \\ x_2(\tau_k^+) = x_2(\tau_k), \end{cases} \tag{5.1}$$

has no nontrivial solution in $PC_\omega \times PC_\omega$. In this case, the nonhomogeneous system

$$\begin{cases} y_1' = c_{11}y_1(t) + c_{12}y_2(t) + f_1, & t \neq \tau_k \\ y_2' = c_{22}y_2(t) + f_2, & k = 0, 1, 2 \cdots. \\ y_1(\tau_k^+) = (1 - p_k)y_1(\tau_k), & t = \tau_k \\ y_2(\tau_k^+) = y_2(\tau_k), \end{cases} \tag{5.2}$$

has a unique solution $(y_1, y_2) \in PC_\omega \times PC_\omega$ for every $(f_1, f_2) \in PC_\omega \times PC_\omega$ and the operator $L : PC_\omega \times PC_\omega \to PC_\omega \times PC_\omega$ defined by $(y_1, y_2) = L(f_1, f_2)$ is linear and relatively compact.

(2). if $\bar{c}_{11} \neq \frac{1}{\omega} \ln[\prod_{k=1}^q \frac{1}{1-p_k}]$ and $\bar{c}_{22} = 0$, then (5.1) has exactly one independent solution in $PC_\omega \times PC_\omega$.

Proof. (1) Since

$$x_2(t) = x_2(0) \exp \int_0^t c_{22}(s)ds \tag{5.3}$$

the condition $\bar{c}_{22} \neq 0$ implies that $x_2 \notin PC_\omega$ unless $x_2 \equiv 0$. Then $x_1(t) = \prod_{0 < \tau_k < t}(1 - p_k)x_1(0) \exp \int_0^t c_{11}(s)ds$ and $\bar{c}_{11} \neq \frac{1}{\omega} \ln[\prod_{k=1}^q \frac{1}{1-p_k}]$ in turn implies $x_1 \notin PC_\omega$ unless $x_1 \equiv 0$.

In this case,

$$y_2' = c_{22}y_2(t) + f_2$$

has a unique solution $y_2(t) \in PC_\omega$ and the operator $L_2 : PC_\omega \to PC_\omega$ defined by $y_2 \equiv L_2 f$ is linear and relatively compact. Furthermore,

$$\begin{cases} y_1' = c_{11}y_1(t) + f_3, & t \neq \tau_k, \\ y_1(\tau_k^+) = (1 - p_k)y_1(\tau_k), & t = \tau_k, \end{cases}$$

for $f_3 = c_{12}L_2f_2 + f_1 \in PC_\omega$ has a unique solution (since $\bar{c}_{11} \neq \frac{1}{\omega}\ln[\prod\limits_{k=1}^{q} \frac{1}{1-p_k}]$) in PC_ω and $y_1 = L_1 f_3$ defines a linear, relatively compact operator

$$L_1 : PC_\omega \to PC_\omega.$$

Thus (5.2) has a unique ω-period solution in $PC_\omega \times PC_\omega$ given by $(y_1, y_2) = L(f_1, f_2)$, where

$$L(f_1, f_2) = (L_1(c_{12}L_2f_2) + f_1), L_2f_2). \tag{5.4}$$

(2). Under the stated assumptions, x_2 as given in (5.3) lies in PC_ω for all initial conditions $x_2(0)$. Now if $\bar{c}_{11} \neq \frac{1}{\omega}\ln[\prod\limits_{k=1}^{q} \frac{1}{1-p_k}]$, then

$$\begin{cases} x_1' = c_{11}y_1(t) + c_{12}x_2(0)exp\int_0^t c_{22}(s)ds, & t \neq \tau_k, \\ x_1(\tau_k^+) = (1 - p_k)x_1(\tau_k), & t = \tau_k, \end{cases}$$

has a unique solution in PC_ω.

Lemma 5.2.[18] Suppose $a \in PC_\omega$ and $\bar{a} = \frac{1}{\omega}\ln[\prod\limits_{k=1}^{q} \frac{1}{1+c_k}]$, then

$$\begin{cases} z' = az + f, & t \neq \tau_k, \\ z(\tau_k^+) = (1 + c_k)z(\tau_k), & t = \tau_k, \end{cases} \tag{5.5}$$

has a solution $z \in PC_\omega$ if and only if

$$\int_0^\omega \prod_{0<\tau_k<t} \frac{1}{1 + c_k} \exp[-\int_0^t a(u)du]f dt = 0$$

For the existence of positive periodic solution, we have the following theorem:

Theorem 5.1. Let us assume $\beta(t) \in PC_\omega$, then there exists a sufficiently small constant $c_0 > 0$, such that for each $\beta(t) \in PC_\omega$ satisfying $b + \alpha + \gamma < \overline{\beta(t)S^*(t)} < b + \alpha + \gamma + c_0$, there exists a solution $(S(t), I(t)) \in PC_\omega \times PC_\omega$ of (2.3) and (2.4) satisfying $S(t) < S^*(t), I(t) > 0$ for all t.

Proof. Let $u_1 = S(t) - S^*(t), u_2 = I(t)$ in (2.3) and (2.4), then we have

$$\begin{cases} u_1' = -bu_1 - \beta(t)S^*(t)u_2 - \beta(t)u_1u_2, & t \neq \tau_k \\ u_2' = [\beta(t)S^*(t) - (b + \gamma + \alpha)]u_2 + \beta(t)u_1u_2, & k = 0, 1, 2 \cdots. \\ u_1(\tau_k^+) = (1 - p_k)u_1(\tau_k), & t = \tau_k \\ u_2(\tau_k^+) = u_2(\tau_k), \end{cases} \tag{5.6}$$

Denote $\theta(t) = \beta(t)S^*(t) - \mu, \mu = \overline{\beta(t)S^*(t)}$, then $\beta(t)S^*(t) = \theta(t) + \mu$ and $\overline{\theta(t)} = 0$, thus (5.6) becomes

$$\begin{cases} u_1' = -bu_1 - \beta(t)S^*(t)u_2 - \beta(t)u_1u_2, & t \neq \tau_k \\ u_2' = [\theta(t) - (b + \gamma + \alpha)]u_2 + \mu u_2 + \beta(t)u_1u_2, & k = 0, 1, 2 \cdots. \\ u_1(\tau_k^+) = (1 - p_k)u_1(\tau_k), & t = \tau_k \\ u_2(\tau_k^+) = u_2(\tau_k). \end{cases} \tag{5.7}$$

Consider the linear homogeneous system

$$\begin{cases} v_1' = -bv_1 - \beta(t)S^*(t)v_2, & t \neq \tau_k \\ v_2' = [\theta(t) - (b + \gamma + \alpha)]v_2, & k = 0, 1, 2 \cdots. \\ v_1(\tau_k^+) = (1 - p_k)v_1(\tau_k), & t = \tau_k \\ v_2(\tau_k^+) = v_2(\tau_k). \end{cases} \tag{5.8}$$

Notice that

$$\overline{c_{22}} = \overline{\theta(t) - (b + \gamma + \alpha)} = -(b + \gamma + \alpha) \neq 0,$$
$$\overline{c_{11}} = -b \neq \frac{1}{\omega} \ln\left[\prod_{k=1}^{q} \frac{1}{1 - p_k}\right] > 0,$$

and so (5.8) satisfies the hypotheses of part (1) of Lemma 5.1. Consequently we have the compact linear operator $L : PC_\omega \times PC_\omega \to PC_\omega \times PC_\omega$ given by (5.4). Using L we can equivalently write system (5.7) as the operator equation

$$(u_1, u_2) = \mu L^*(u_1, u_2) + G(u_1, u_2), \tag{5.9}$$

where

$$L^*(u_1, u_2) = (L_1(-\beta(t)S^*(t)L_2 u_2), L_2 u_2),$$
$$G(u_1, u_2) = (L_1(-\beta(t)S^*(t)L_2 g_2(u_1, u_2) + g_1(u_1, u_2)), L_2 g_2),$$

and $g_1(u_1, u_2) = -\beta(t)u_1 u_2$, $g_2(u_1, u_2) = \beta(t)u_1 u_2$. Here $L : PC_\omega \times PC_\omega \to PC_\omega \times PC_\omega$ is linear and relatively compact, and $G : PC_\omega \times PC_\omega \to PC_\omega \times PC_\omega$ is quasiequicontinuous and relatively compact and satisfies $G = o(\| (u_1, u_2) \|_{PC_\omega})$ near $(0, 0)$. Note that we have already got the operator equation (5.9) in such a form so that now we can apply standard bifurcation theorems and techniques directly to this equation. A non-trivial solution $(u_1, u_2) \neq (0, 0)$ of (5.9) in $PC_\omega \times PC_\omega$, for some $\mu \in R$ (R is the set of the reals), yields a solution $(S, I) = (u_1 + S^*, u_2)$ of the system (2.3) and (2.4) for $\mu = \overline{\beta(t)S^*(t)}$. Solution $(S, I) \neq (S^*, 0)$ will be called non-trivial solutions of (2.3) and (2.4).

To prove Theorem 5.1, we are going to apply some well-known but pretty standard local bifurcation techniques to the operator equations (5.9)[17]. The bifurcation can occur only at the non-trivial solutions of the linearized problem

$$(v_1, v_2) = \mu L^*(v_1, v_2), \quad (v_1, v_2) \neq (0, 0), \quad \mu \in R. \tag{5.10}$$

Let $(v_1, v_2) \in PC_\omega \times PC_\omega$ be a solution of equation (5.10) for some $\mu \in R$, then by the very manner in which L^* is defined, (v_1, v_2) satisfies the system

$$\begin{cases} v_1' = -bv_1 - \beta(t)S^*(t)v_2, & t \neq \tau_k \\ v_2' = [\theta(t) + \mu - (b + \gamma + \alpha)]v_2, & k = 0, 1, 2 \cdots . \\ v_1(\tau_k^+) = (1 - p_k)v_1(\tau_k), & t = \tau_k \\ v_2(\tau_k^+) = v_2(\tau_k). \end{cases} \tag{5.11}$$

and conversely. By using Lemma 5.1, we see that (5.11) and hence (5.10) has a non-trivial solution in $PC_\omega \times PC_\omega$ if and only if $\mu = \mu^*$ where

$$\mu^* = b + \gamma + \alpha,$$

If $\mu = \mu^*$, then, by part (2) of Lemma 5.1, (5.11) has one independent solution in $PC_\omega \times PC_\omega$. Therefore a well-known result [17, 20-21] says that bifurcation occurs at this simple eigenvalue; hence there exists a continuum $U = \{(\mu, u_1, u_2)\} \subseteq R \times PC_\omega \times PC_\omega$ of non-trivial solutions of (5.9) such that the closure $\overline{U}$ of U contains $(\mu^*, 0, 0)$. This continuum gives rise to a continuum $C = \{(\mu; S, I)\} \subseteq R \times PC_\omega \times PC_\omega$ of non-trivial solutions of (2.3) and (2.4) whose closure $\overline{C}$ contains the bifurcation point $(\mu^*; S^*, 0)$.

To observe that particular solutions in C correspond to solutions (S, I) of (2.3) and (2.4) with the properties described in Theorem 3.1, we investigate the nature of the continuum C near the bifurcation point $(\mu^*, 0, 0)$ by expanding μ and (u_1, u_2) in the Lyapunov-Schmidt series (for small ε) [22]:

$$\mu = \mu^* + \mu_1 \varepsilon + \cdots, \quad u_i = u_{i1}\varepsilon + u_{i2}\varepsilon^2 + \cdots, i = 1, 2$$

for $u_{ij} \in PC_\omega$. We substitute these series into the system (5.7) and equate the coefficients of ε and ε^2, we find

$$\begin{cases} u_{11}' = -bu_{11} - \beta(t)S^*(t)u_{21}, & t \neq \tau_k \\ u_{21}' = [\theta(t) + \mu^* - (b + \gamma + \alpha)]u_{21}, & k = 0, 1, 2 \cdots . \\ u_{11}(\tau_k^+) = (1 - p_k)u_{11}(\tau_k), & t = \tau_k \\ u_{21}(\tau_k^+) = u_{21}(\tau_k). \end{cases} \tag{5.12}$$

and

$$\begin{cases} u'_{12} = -bu_{12} - \beta(t)S^*(t)u_{22} - \beta(t)u_{11}u_{21}, & t \neq \tau_k \\ u'_{22} = [\theta(t) + \mu^* - (b + \gamma + \alpha)]u_{22} + \mu_1 u_{21} + \beta(t)u_{11}u_{21}, & \\ u_{12}(\tau_k^+) = (1 - p_k)u_{12}(\tau_k), & t = \tau_k \\ u_{22}(\tau_k^+) = u_{22}(\tau_k), \ k = 0, 1, 2 \cdots . \end{cases} \tag{5.13}$$

Obviously $(u_{11}, u_{21}) \in PC_\omega \times PC_\omega$ must be a solution of (5.11). We choose the specific solution satisfying the initial conditions $u_{21}(0) = 1$. Then

$$u_{21}(t) = \exp(\int_0^t [\theta(s) + \mu^* - (b + \gamma + \alpha)]ds) > 0. \tag{5.14}$$

Moreover, $u_{11}(t)$ is the ω-periodic solution of the linear equation

$$\begin{cases} u'_{11} = -bu_{11} - \beta(t)S^*(t)u_{21}, \ t \neq \tau_k \\ u_{11}(\tau_k^+) = (1 - p_k)u_{11}(\tau_k), \quad t = \tau_k \end{cases} \tag{5.15}$$

Hence

$$u_{11} = -\int_0^\omega G(t, s)\beta(s)S^*(s)u_{21}(s)ds,$$

Where $G(t, s)$ is Green's function given by

$$G(t, s) = \begin{cases} X(t)(1 - X(\omega))^{-1}/X(s), & 0 \leq s \leq t \leq \omega, \\ X(t + \omega)(1 - X(\omega))^{-1}/X(s), & 0 \leq t \leq s \leq \omega, \\ G(t - k\omega, s - j\omega), & j\omega < s < j\omega, k\omega < t < k\omega + \omega, \end{cases}$$

and

$$X(t) = \prod_{0 < \tau_k < t} (1 - p_k)e^{-bt}.$$

Note that $u_{12}(t) > 0$ for all t, and also since

$$X(\omega) = \prod_{k=1}^q (1 - p_k)e^{-b\omega} < 1,$$

therefore Green's function $G(t, s) > 0$ and thus $u_{11}(t) < 0$ for all t. Using Lemma 5.2 on the second equation of (5.13), we obtain

$$\int_0^\omega u_{21}(t)(\mu_1 + \beta(t)u_{11}) \exp(\int_0^t [\theta(s) + \mu^* - (b + \gamma + \alpha)]ds) = 0. \tag{5.16}$$

From the equation of (5.16), we obtain

$$\mu_1 = -\overline{\beta(t)u_{11}(t)} > 0.$$

Thus we conclude (see [17]) that near the bifurcation point $(\mu^*; 0, 0)$ (say, for a sufficiently small constant $c_0 > 0$, $\mu^* < \mu < \mu^* + c_0$), there is a branch $U^+ = \{(\mu; u_1, u_2) \in U : \mu^* \leq \mu < \mu^* + c_0, u_1(t) < 0, u_2(t) > 0\}$.

Therefore, we have the piecewise continuous branch C^+ of solutions of the from $(\mu; S, I) \in R \times PC_\omega \times PC_\omega$, satisfying $S(t) < S^*(t), I(t) > 0$ for all t. Since $\mu^* \leq \mu < \mu^* + c_0$ is equivalent to $b + \alpha + \gamma < \overline{\beta(t)S^*(t)} < b + \alpha + \gamma + c_0$, which again equivalent to $R_0 \to 1^+$. Hence the proof.

6. Numerical Simulation

In our proposed pulse vaccination SIR model with periodic infection rate $\beta(t)$ we obtain that the periodic infection-free solution is globally stable provided the basic reproductive number R_0 is less than 1. On the other hand, periodic infection-free solution is unstable if the basic reproductive number is greater than 1 and in this case the disease will persist in the population. Moreover, the positive periodic solution exists when $R_0 \to 1^+$.

In this section we have performed numerical simulation to show the geometric impression of our results. In all simulations, the units are set to unity (scaled to unity).
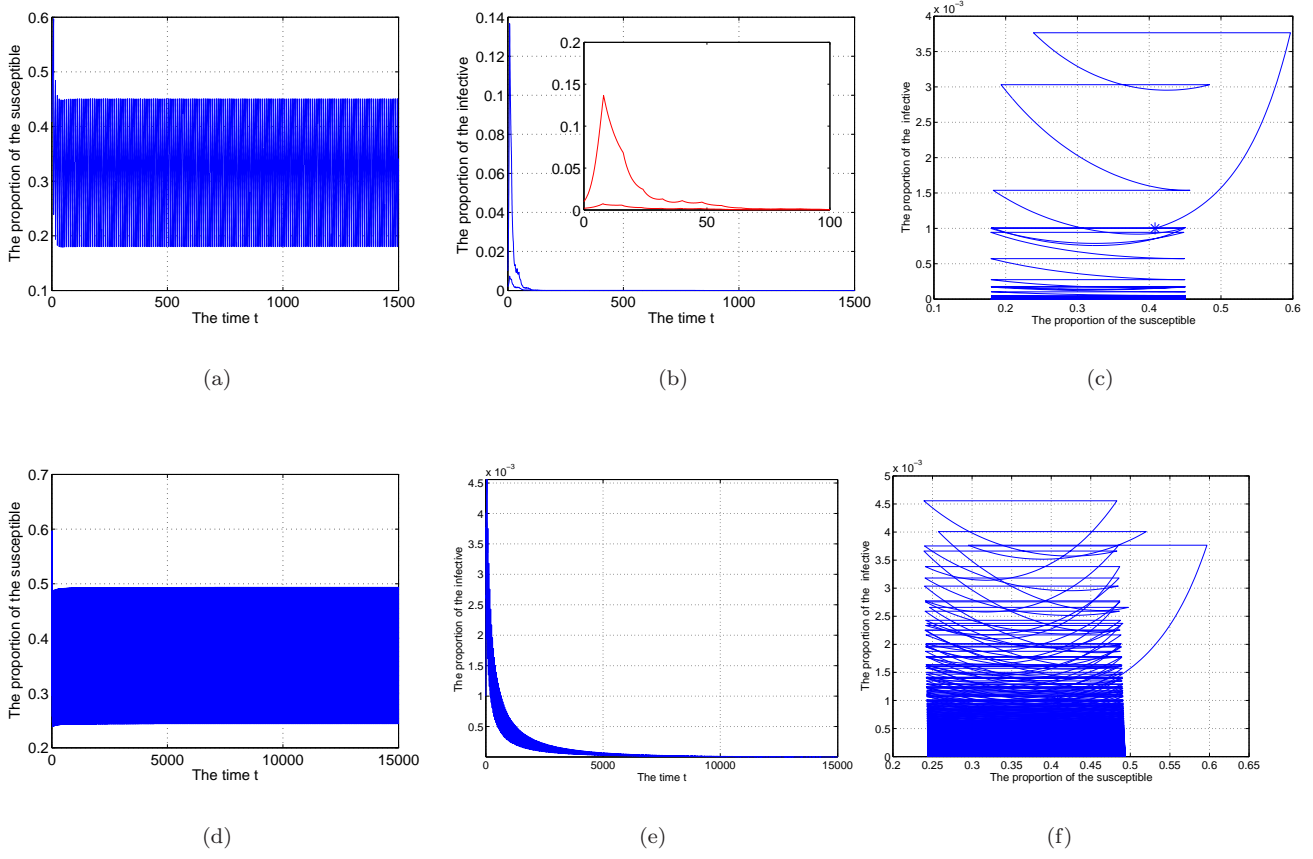

(a)   (b)   (c)

(d)   (e)   (f)

FIG. 1: Solution of system (2.3) and (2.4) with $R_0 < 1$. Subfigures 1(a),(d), and (b), (e) demonstrate respectively the susceptible and infective population with respect to time, and Figure 1 (c), (f) show their corresponding phase-portrait i.e in $S - I$ plane.

To demonstrate the global stability of the system (2.3) and (2.4) we take following set parameter values: $b = 0.05$, $K = 1$, $\gamma = 0.25$, $\alpha = 0.002$ and $\beta(t) = 0.8[1 + 0.2\cos(0.05\pi t)]$. Figure 1 depicts the global stability of system (2.3) and (2.4) when $R_0 < 1$ with initial conditions $S(0) = 0.408166$, $I(0) = 0.002$, $R(0) = 0.08$. Taking a fixed value of $\tau_k = 8$ we have varied value the parameter p_k. Fig. 1(a)–(c) shows the solution for $p_k = 0.6$, $R_0 = 0.85909$ and for Fig. 1(d)-(f) we take $p_k = 0.505$, $R_0 = 0.99892$ ($R_0 \to 1^-$).
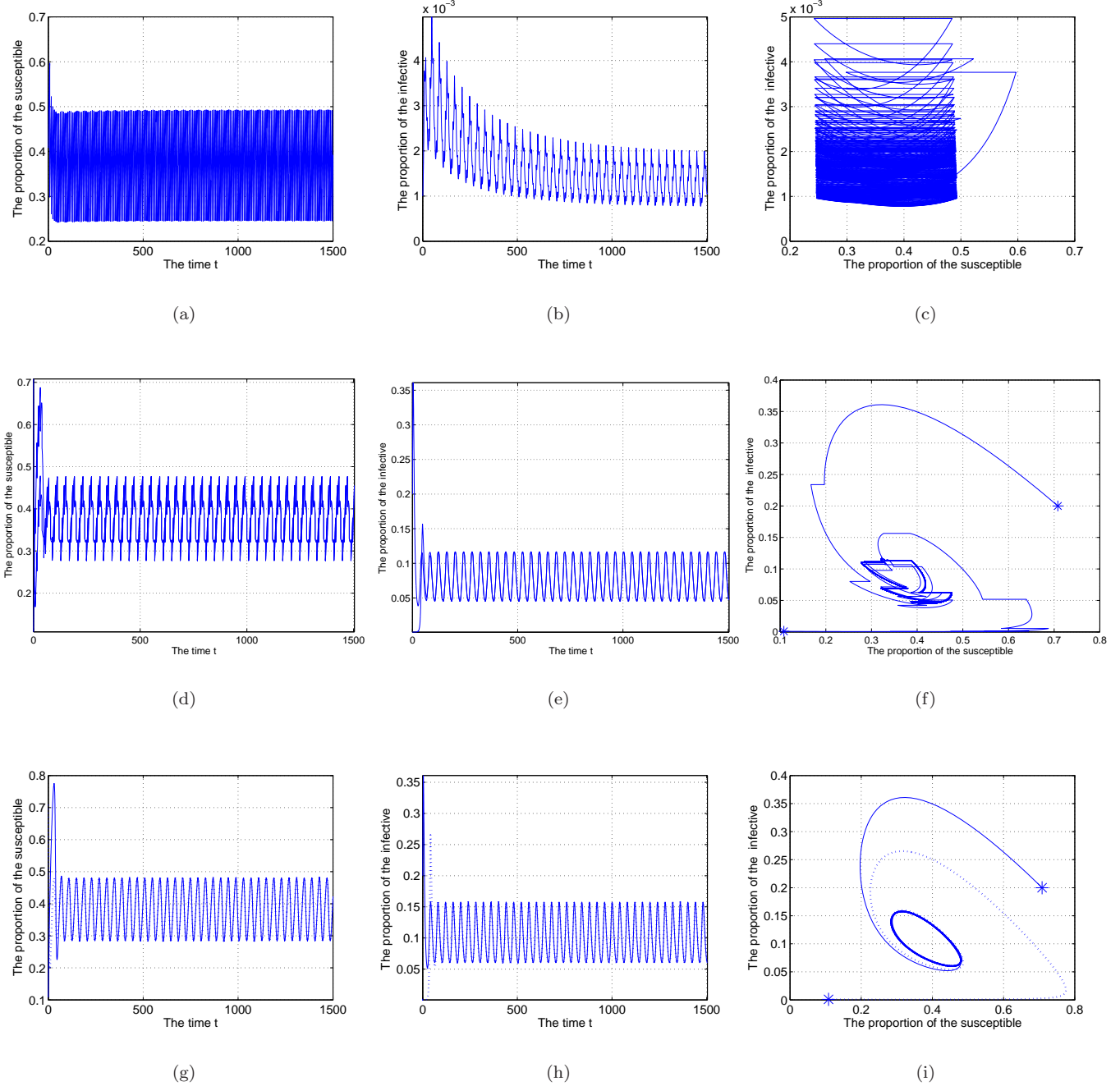
FIG. 2: Unstable infection-free situation of the system (2.3) and (2.4) at $R_0 > 1$. Subfigure 2(a),(d),(g) and (b),(e),(h) demonstrate respectively the susceptible and infective population with respect to time t. Subfigure 2(c),(f) and (i) show their respective phase portrait i.e in $S - I$ plane.

By choosing $R_0 > 1$ we have shown the unstable infection free solution of the system (2.3) and (2.4) in figure 2. Here we have chosen two sets of initial conditions: $S(0) = 0.108166$, $I(0) = 0.001$, $R(0) = 0.08$ and $S(0) = 0.708166$, $I(0) = 0.2$, $R(0) = 0.08$. The Fig. 2(a)–(c) with $p_k = 0.5$ and $R_0 = 1.00702$; The Fig. 2(d)–(f) with $p_k = 0.15$ and $R_0 = 1.88779$; The Fig. 2(g)–(i) is the limiting case with $p_k = 0$ and $R_0 = 2.64901$. The other parameters have the same value as in Fig. 1.

Now we fix the parameter p_k and vary the parameter τ_k. In figure 3, the initial values are used $S(0) = 0.408166$, $I(0) = 0.002$, $R(0) = 0.08$. Fig. 3(a)–(c) the parameters are chosen as $p = 0.5$ and $\tau = 5$ with $R_0 = 0.72972$;

Fig. 3(d)–(f) the parameters are chosen as $p = 0.5$ and $\tau_k = 13$ with $R_0 = 1.34774$, the other parameters as the same Fig.1. Fig. 3 suggests the positive periodic solution is globally asymptotically stable when the $R_0 > 1$.
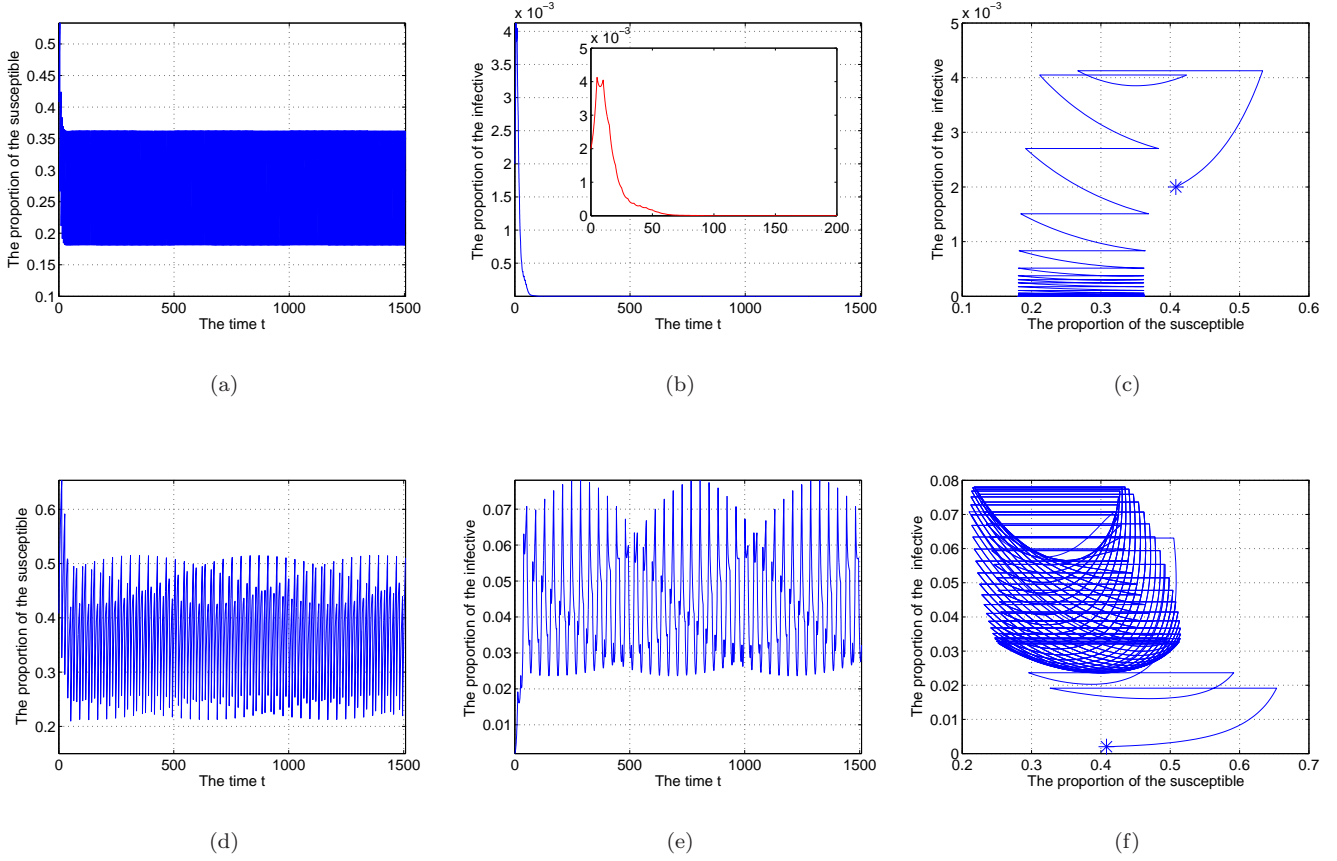

(a)   (b)   (c)

(d)   (e)   (f)

FIG. 3: The orbits of the system (2.3) and (2.4)with respect to time series. Subfigure 3(c), (f) depict the orbits on $S - I$ plane, and (a), (d) and (b), (e) show the time series for susceptible and infective respectively.

7. Discussion

Vaccination strategies are designed and applied to control or eradicate an infection from the population. As for many directly transmitted infectious diseases levels of infection oscillate regularly it may be more useful to use a time dependent contact rate in place of a constant contact rate.

The idea of this vaccination strategy is to give the whole susceptible population vaccine at a periodically varying rate $\beta(t)$: Many vaccination strategies are applied in different parts of the world. Some of these use a periodic vaccination strategy. The sort of periodic vaccination rate most commonly used is the pulse periodic function [24]. As the study of periodicity and other oscillatory behavior in the incidence of infectious diseases, especially childhood diseases, is one of our main goals in this work, we shall study the effect of a general time dependent periodic vaccination strategy on the dynamics of these infectious diseases. This new vaccination strategy is designed to vaccinate susceptible of all ages using a varying periodic vaccination rate $\beta(t)$. This periodic vaccination is intended to keep the disease free solution stable by controlling the number of susceptible over the vaccination interval. We have study an SIR epidemic model with seasonality in the transmission rate but the results apply equally well to the case where the contact rate is constant.

A lot of infectious diseases have, at least, temporary varying transmission rate due to the seasonal fluctuation or changing social behavior. The time dependent epidemic models with pulse vaccination are natural generalization and have more applications in real world. The study of the dynamical behaviors of epidemic models with time dependent infection rate and pulse vaccination is an important and challenging issue for us. The SIR epidemic model with periodic infection rate and pulse vaccination is one of the simple and important epidemic models. In this article, we have investigated the dynamical behavior of a classical periodic SIR model with pulse vaccination. We have shown that the infection–free periodic solution $(S^*(t), 0, R^*(t))$ is globally stable if the basic reproductive number R_0 is less than 1,

and the infection–free periodic solution $(S^*(t), 0, R^*(t))$ is unstable if the basic reproductive number R_0 is greater than 1. In the later case, the disease will uniform persist in the population. Moreover, we have used a standard bifurcation technique to show the existence of the positive periodic solutions which arise near the infection–free periodic solution $(S^*(t), 0, R^*(t))$.

The basic reproductive number is a fundamental parameter and one of the most useful quantities characterize the magnitude of an infectious disease transmission [23]. It is the average number of secondary cases that arise from a primary case in a fully susceptible population. For the classical SIR model with constant infection rate, the basic reproductive number R_0 is the product of the average infection period and the average infection in unit time. From a deterministic viewpoint, the eradication of an infection requires $R_0 < 1$, and the condition for outbreak or for maintenance of epidemic infection demands $R_0 > 1$. The basic reproductive number R_0 for the SIR model with periodic infection rate and pulse vaccination is defined using the average of $\beta(t)S^*(t)$ over one period. It is proved that the disease will die out if $R_0 < 1$, and the disease will persist when $R_0 > 1$. The classical SIR model has a globally stable endemic equilibrium. The periodic solution of the SIR model with periodic infection rate and pulse vaccination is also expected to play the similar role. However, here we have got comparatively few new results on the existence of the positive periodic solution of the epidemic models. The bifurcation theorem has been used to establish the existence of periodic solution and found that positive periodic solution of our system is globally stable when $R_0 > 1$. On the other hand, infection free periodic solution is globally stable if $R_0 < 1$. We have simulated the global behaviors of the model for both cases. From the numerical simulation, we conclude that the positive periodic solution of the SIR model is unique and globally stable. The uniqueness and stability of the periodic solution is of course more challenging problem for the researcher [24]. We expect the uniqueness and global stability of the periodic solution will be proved very soon.

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